

Chapter 4

Forward Differencing

Horner's algorithm is the fastest method for evaluating a polynomial at a single point. For a degree n polynomial, it requires n multiplies and n adds.

If a polynomial is to be evaluated at several evenly spaced values $t, t + \delta, t + 2\delta, \dots, t + k\delta$, the fastest method is to use *forward differences*.

Consider a degree 1 polynomial

$$f(t) = a_0 + a_1t.$$

The difference between two adjacent function values is

$$\Delta_1(t) = f(t + \delta) - f(t) = [a_0 + a_1(t + \delta)] - [a_0 + a_1t] = a_1\delta.$$

Thus, $f(t)$ can be evaluated at several evenly spaced points using the algorithm:

```
 $\Delta_1 = a_1\delta$   
 $t_0 = 0$   
 $f(0) = a_0$   
for  $i = 1$  to  $k$  do  
   $t_i = t_{i-1} + \delta$   
   $f(t_i) = f(t_{i-1}) + \Delta_1$   
endfor
```

Thus, each successive evaluation requires only one add, as opposed to one add and one multiply for Horner's algorithm.

This idea extends to polynomials of any degree. For the quadratic case,

$$f(t) = a_0 + a_1t + a_2t^2.$$

The difference between two adjacent function values is

$$\Delta_1(t) = f(t + \delta) - f(t) = [a_0 + a_1(t + \delta) + a_2(t + \delta)^2] - [a_0 + a_1t + a_2t^2]$$

$$\Delta_1(t) = a_1\delta + a_2\delta^2 + 2a_2t\delta.$$

We can now write

```
t0 = 0
f(0) = a0
for i = 1 to k do
  ti = ti-1 + δ
  Δ1(ti) = a1δ + a2δ2 + 2a2ti-1δ
  f(ti) = f(ti-1) + Δ1(ti-1)
endfor
```

In this case, $\Delta(t)$ is a linear polynomial, so we can evaluate it as above, by defining

$$\Delta_2(t) = \Delta_1(t + \delta) - \Delta_1(t) = 2a_2\delta^2$$

and our algorithm now becomes

```
t0 = 0
f(0) = a0
Δ1 = a1δ + a2δ2
Δ2 = 2a2δ2
for i = 1 to k do
  ti = ti-1 + δ
  f(ti) = f(ti-1) + Δ1
  Δ1 = Δ1 + Δ2
endfor
```

It should be clear that for a degree n polynomial, each successive evaluation requires n adds and no multiplies! For a cubic polynomial

$$f(t) = a_0 + a_1t + a_2t^2 + a_3t^3,$$

the forward difference algorithm becomes

```
t0 = 0
f(0) = a0
Δ1 = a1δ + a2δ2 + a3δ3
Δ2 = 2a2δ2 + 6a3δ3
Δ3 = 6a3δ3
for i = 1 to k do
  ti = ti-1 + δ
  f(ti) = f(ti-1) + Δ1
  Δ1 = Δ1 + Δ2
  Δ2 = Δ2 + Δ3
endfor
```

Several questions remain. First, what are the initial values for the Δ_i if we want to start at some value other than $t = 0$. Second, what is a general equation for the Δ_i for a general degree n polynomial $f(t)$. Also, what if our polynomial is not in power basis.

These questions can be answered almost trivially by observing the following. Since $t_{i+1} = t_i + \delta$, we have

$$\begin{aligned}\Delta_1(t_i) &= f(t_{i+1}) - f(t_i); \\ \Delta_j(t_i) &= \Delta_{j-1}(t_{i+1}) - \Delta_{j-1}(t_i), \quad j = 2, \dots, n; \\ \Delta_n(t_i) &= \Delta_n(t_{i+1}) = \Delta_n(t_{i+k}) = \text{a constant} \\ \Delta_{n+1} &= 0\end{aligned}$$

Thus, our initial values for $\Delta_j(t_i)$ can be found by simply computing $f(t_i), f(t_{i+1}), \dots, f(t_{i+n})$ and from them computing the initial differences. This lends itself nicely to a table. Here is the table for a degree four case:

$f(t_i)$	$f(t_{i+1})$	$f(t_{i+2})$	$f(t_{i+3})$	$f(t_{i+4})$
$\Delta_1(t_i)$	$\Delta_1(t_{i+1})$	$\Delta_1(t_{i+2})$	$\Delta_1(t_{i+3})$	
$\Delta_2(t_i)$	$\Delta_2(t_{i+1})$	$\Delta_2(t_{i+2})$		
$\Delta_3(t_i)$	$\Delta_3(t_{i+1})$			
$\Delta_4(t_i)$				
0	0	0	0	0

To compute $f(t_{i+5})$, we simply note that every number R in this table, along with its right hand neighbor R_{right} and the number directly beneath it R_{down} obey the rule

$$R_{right} = R + R_{down}.$$

Thus, we can simply fill in the values

$$\begin{aligned}\Delta_4(t_{i+1}) &= \Delta_4(t_i) + 0 \\ \Delta_3(t_{i+2}) &= \Delta_3(t_{i+1}) + \Delta_4(t_{i+1}) \\ \Delta_2(t_{i+3}) &= \Delta_2(t_{i+2}) + \Delta_3(t_{i+2}) \\ \Delta_1(t_{i+4}) &= \Delta_1(t_{i+3}) + \Delta_2(t_{i+3}) \\ f(t_{i+5}) &= f(t_{i+4}) + \Delta_1(t_{i+4})\end{aligned}$$

Note that this technique is independent of the basis in which $f(t)$ is defined. Thus, even if it is defined in Bernstein basis, all we need to do is to evaluate it $n + 1$ times to initiate the forward differencing.

For example, consider the degree 4 polynomial for which $f(t_i) = 1, f(t_{i+1}) = 3, f(t_{i+2}) = 2, f(t_{i+3}) = 5, f(t_{i+4}) = 4$. We can compute $f(t_{i+5}) = -24, f(t_{i+6}) = -117$, and $f(t_{i+7}) = -328$ from the following difference table:

$t :$	t_i	t_{i+1}	t_{i+2}	t_{i+3}	t_{i+4}	t_{i+5}	t_{i+6}	t_{i+7}
$f(t) :$	1	3	2	5	4	-24	-117	-328
$\Delta_1(t) :$	2	-1	3	-1	-28	-93	-211	
$\Delta_2(t) :$	-3	4	-4	-27	-65	-118		
$\Delta_3(t) :$	7	-8	-23	-38	-53			
$\Delta_4(t) :$	-15	-15	-15	-15	-15			
$\Delta_5(t) :$	0	0	0	0	0	0	0	0

