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Improvement on the Dimensions of Spline Spaces on T-Mesh ⋆

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Abstract

Deng et al. firstly proposed a method based on B-nets to calculate the dimension of a spline space $\mathcal{S}(m, n, \alpha, \beta, \mathcal{T})$ over a T-mesh with constraint $m \geq 2\alpha + 1$ and $n \geq 2\beta + 1$. In this paper, we discuss the dimensions of the same spline spaces on the T-mesh by using the Smoothing Cofactor-Conformality method. The result is improved essentially with another relaxed constraint depending on the order of the smoothness, the degree of the spline functions and the structure of the T-mesh as well.

Keywords: Spline space; Smoothing Cofactor-Conformality method; Dimension formula; T-mesh

1 Introduction

A T-mesh is basically a rectangular grid that allows T-junctions arising from T-spline ([3], [4]). T-meshes are formed by a set of horizontal line segments and a set of vertical line segments. In paper [1], Deng et al. firstly proposed a method based on B-nets to calculate the dimension of a spline space $\mathcal{S}(m, n, \alpha, \beta, \mathcal{T})$ over a regular T-mesh whose boundary grid lines form a rectangle (see Fig. 1(a)). When the smoothness is less than half of the degree of the spline functions (i.e., $m \geq 2\alpha + 1$ and $n \geq 2\beta + 1$), a dimension formula is derived which involves only the topological quantities of the T-mesh. However, the constraint is too strong and independent of the structure of the T-mesh. It excludes even the usual tensor product B-spline spaces where $m = \alpha + 1$ and $n = \beta + 1$.

In this paper, we discuss the dimensions of the same spline spaces on the T-mesh by using the Smoothing Cofactor-Conformality method ([5], [6]). The result is improved essentially with another relaxed constraint depending on the order of the smoothness, the degree of the spline functions and the structure of the T-mesh as well. By the constraint conditions, the dimension formula is applicable to the high order of the smoothness and low degree of the spline spaces

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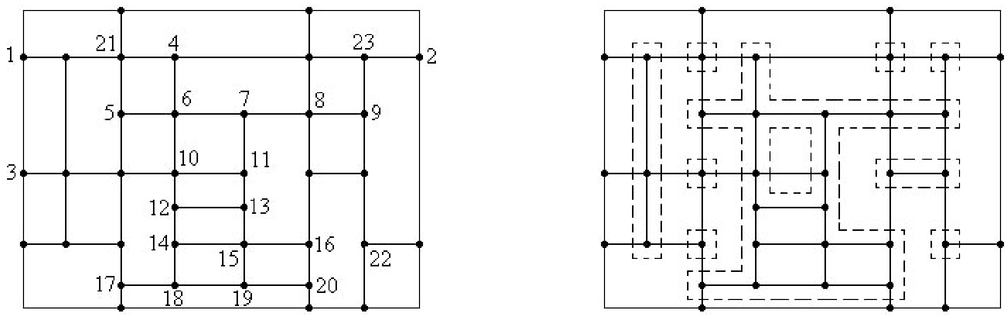
defined on corresponding T-mesh. It admits the spline spaces with desired smoothness and degree by adjusting the topological structure of T-mesh easily.

The rest of the paper is organized as follows. First, we introduce some definitions and notations for T-mesh in Section 2. In Section 3, we present an improved dimension formula and proof by decomposing the T-mesh into some T-connected components. We obtain an intrinsic relation between the degree, the smoothness and the structure of the T-mesh, which guarantees the dimension formula. At last, we prove that the dimension formula is equivalent to the formula given by Deng et al. in [1] when the constraints degenerate to $m \geq 2\alpha + 1$ and $n \geq 2\beta + 1$.

2 Some Definitions and Notations for T-mesh



(a) (b)

Fig. 1: A T-mesh and its T-connected components.

Given a T-mesh (as shown in Fig. 1(a)), there are three kinds of interior mesh segments.

a. *cross-cut*: the two endpoints of the segment lie on the boundary of the T-mesh (e.g. v_1v_2).

b. *ray*: only one endpoint lies on the boundary of the T-mesh (e.g. v_3v_{11}).

c. *T-segment*: both of the endpoints don't lie on the boundary of the T-mesh (e.g. v_4v_{18}, v_5v_9).

According to the three kinds of mesh segments, there are also three kinds of interior vertices.

a. *free-vertex*: the vertex is an intersection point of two cross-cuts or two rays or one cross-cut and one ray (e.g. v_{21}, v_{22}, v_{23}).

b. *mono-vertex*: the vertex is an intersection point of one T-segment and one cross-cut or one ray (e.g. $v_4, v_5, v_9, v_{10}, v_{11}$).

c. *multi-vertex*: the vertex is an intersection point of one horizontal T-segment and one vertical T-segment (e.g. v_6, v_7, v_{12}, v_{13}).

We define a relation between T-segments.

T-connected: one horizontal T-segment and one vertical T-segment are T-connected if they have one common interior vertex. For example, v_5v_9 and v_4v_{18} are T-connected at v_6.

We have some concepts based on "T-connected".

a. *T-connected component*: the union of all T-connected T-segments and their vertices.

b. *Boundary vertex of T-connected component*: the vertex is an endpoint of one T-segment, which lies on one cross-cut or one ray (e.g. $v_4, v_5, v_9, v_{16}, v_{17}, v_{20}$).

c. *Interior vertex of T-connected component*: each vertex of the T-connected component except the boundary vertex of T-connected component (e.g. v_6, v_8, v_{13}, v_{15}).

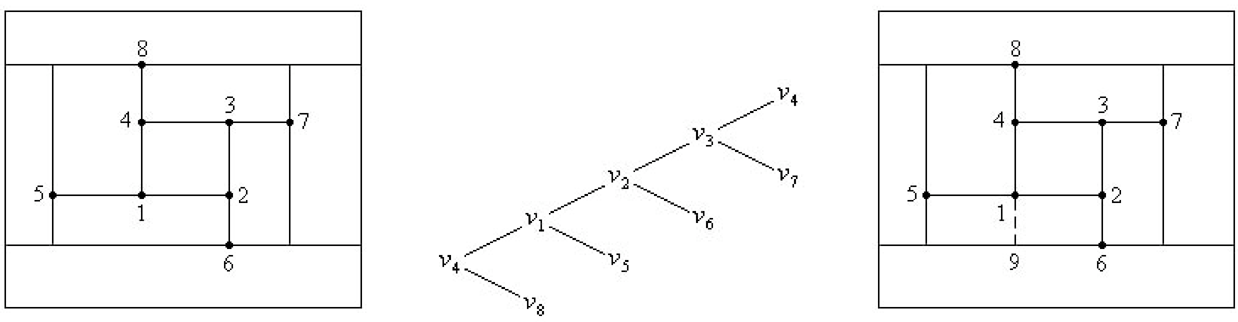
It is clear that the vertices of a T-connected component including at least one T-segment are mono-vertices or multi-vertices, and the boundary vertex of the T-connected component are mono-vertices. Especially, a free-vertex is a single T-connected component (e.g. v_{21}, v_{22}, v_{23}). By the definition, the whole T-mesh can be decomposed into many different T-connected components without intersections (as every independent part closed by dotted line shown in Fig. 1(b)).

We define a special order of T-connected T-segments in a T-connected component.

Regular order of T-connected T-segments: for k T-segments $L_1, \cdots, L_k$ in a T-connected component, if they can be arranged in an order as

$$L_1 \to L_2 \to \cdots \to L_{k-1} \to L_k,$$

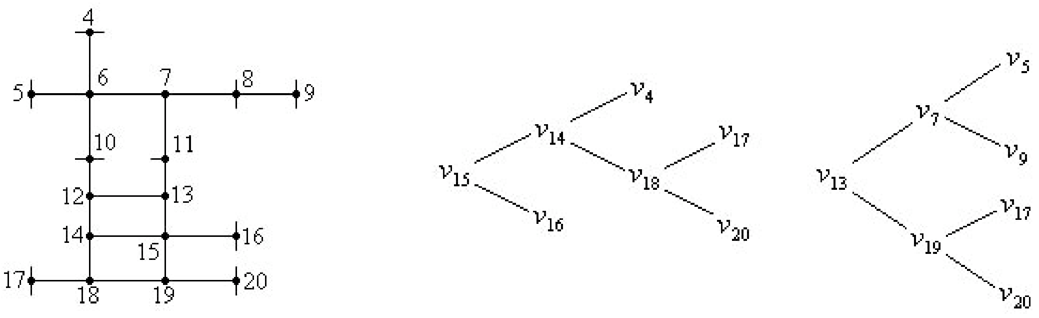
such that the two end vertices of L_1 are boundary vertices of the T-connected component, the two end vertices of L_i are either boundary vertices of the T-connected component or the interior vertices of one of the former $i-1$ T-segments $L_1, \cdots, L_{i-1}$, $i = 2, \cdots, k$. Then we say $L_1, \cdots, L_k$ have a regular order.



(a) (b) (c)

Fig. 2: A T-cycle with its binary tree and its modification.

However, the T-segments cannot be arranged in a regular order for some cases. We refer to a kind of binary tree similar to that in [1] to classify T-connected components. Given a T-connected component, selecting an interior vertex as root, its two child nodes are the two end vertices of the T-segment where the root lies (If it is an intersection point of two T-segments, then it can be regarded as two roots of two different trees). If the node is a boundary vertex of the T-connected component, it has no child. If a node come back to the root in this procedure, then we say the component of the tree is a *T-cycle* (as shown in Fig. 2(a) and 2(b), $v_1v_8 - v_2v_5 - v_3v_6 - v_4v_7$ is a T-cycle). Otherwise, this procedure will stop and we obtain a binary tree, where the leaf nodes are boundary vertices of the T-connected component. Besides, a T-cycle can be modified by adding a virtual mono-vertex as its end vertex (as shown in Fig. 2(c)), then the new T-connected component has no T-cycle.



(a) (b)

Fig. 3: A T-connected component and its binary trees.

By the binary trees, all T-segments in a T-connected component without T-cycle have a regular order. In fact, it can be done easily by the inverse of the binary trees of vertices. Of course, the order is not unique. For example, the 6 T-segments in the T-connected component shown in Fig. 3(a) can be arranged in the following regular order by the inverse of the corresponding binary trees shown in Fig. 3(b).

$$v_5v_9 \longrightarrow v_{17}v_{20} \longrightarrow v_4v_{18} \longrightarrow v_7v_{19} \longrightarrow v_{14}v_{16} \longrightarrow v_{12}v_{13}.$$

3 The dimensions of spline spaces on T-mesh

Given a T-mesh $\mathcal{T}$. Let $\mathcal{F}$ denote all the cells in $\mathcal{T}$ and Ω denote the region occupied by all the cells in $\mathcal{T}$. A spline space defined over $\mathcal{T}$ is ([1])

$$\mathcal{S}(m,n,\alpha,\beta,\mathcal{T}) := \{s(x,y) \in C^{\alpha,\beta}(\Omega) | s(x,y)|_\phi \in \mathbb{P}_{m,n}, \text{for any } \phi \in \mathcal{F}\}, \tag{1}$$

where $\mathbb{P}_{m,n}$ is the space of all polynomials with bi-degree (m,n), and $C^{\alpha,\beta}(\Omega)$ the space consisting of all bivariate functions which are continuous in Ω with order α along x direction and with order β along y direction.

3.1 Two classes of conformality conditions along T-segments

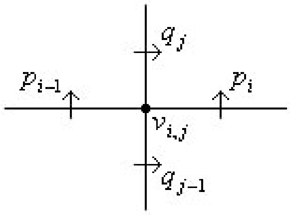
At first, we introduce two classes of conformality conditions on $\mathcal{S}(m,n,\alpha,\beta,\mathcal{T})$. Since the splines in $\mathcal{S}(m,n,\alpha,\beta,\mathcal{T})$ are C^α continuous along x direction and C^β continuous along y direction. By using the Smoothing Cofactor-Conformality method ([5], [6]), there exist $p_i(x,y) \in \mathbb{P}_{m,n-\beta-1}$, the corresponding smoothing cofactors across each horizontal grid segments, and $q_j(x,y) \in \mathbb{P}_{m-\alpha-1,n}$, the corresponding smoothing cofactors across each vertical grid segments.

For each interior vertex (x_i, y_j) denoted by $v_{i,j}$, the conformality condition is (if $v_{i,j}$ is a T-junction, then one of the smoothing cofactors vanishes, as shown in Fig. 4(a))

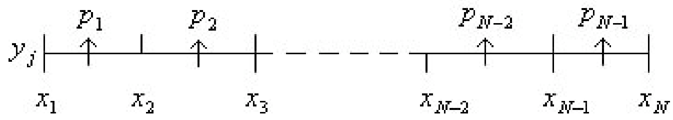
$$(p_i - p_{i-1})(y-y_j)^{\beta+1} + (q_{j-1} - q_j)(x-x_i)^{\alpha+1} = 0. \tag{2}$$

Since $(x-x_i)^{\alpha+1}$ and $(y-y_j)^{\beta+1}$ are prime to each other, there exist $t_i(x,y)$, $t_j(x,y) \in \mathbb{P}_{m-\alpha-1,n-\beta-1}$, such that

$$p_i - p_{i-1} = t_i(x-x_i)^{\alpha+1}, \; q_{j-1} - q_j = t_j(y-y_j)^{\beta+1}.$$



(a)



(b)

Fig. 4: The smoothing cofactors around $v_{i,j}$ and along a horizontal T-segment.

Substituting the above expressions into Eq. (2), we obtain

$$(t_i(x,y) + t_j(x,y))(x - x_i)^{\alpha+1}(y - y_j)^{\beta+1} \equiv 0.$$

Therefore,

$$t_i(x,y) \equiv -t_j(x,y) =: d_{i,j}(x,y) \in \mathbb{P}_{m-\alpha-1,n-\beta-1}.$$

$d_{i,j}(x,y)$ is regarded as the conformality cofactor at interior vertex $v_{i,j}$. Thus, we find quantity relations between smoothing cofactors

$$p_i(x,y) - p_{i-1}(x,y) \equiv d_{i,j}(x,y)(x - x_i)^{\alpha+1}, \tag{3}$$

and

$$q_j(x,y) - q_{j-1}(x,y) \equiv d_{i,j}(x,y)(y - y_j)^{\beta+1}. \tag{4}$$

For a horizontal T-segment including N interior vertices, the corresponding smoothing cofactors $p_i, i = 1, \cdots, N-1$ are shown in Fig. 4(b). We have N conformality conditions at each vertex $v_{i,j}$ as follows.

$$\begin{aligned} p_1(x,y) &= d_{1,j}(x - x_1)^{\alpha+1}; && (5)\\ p_2(x,y) - p_1(x,y) &= d_{2,j}(x - x_2)^{\alpha+1}; && (6)\\ &\cdots && (7)\\ p_{N-1}(x,y) - p_{N-2}(x,y) &= d_{N-1,j}(x,y)(x - x_{N-1})^{\alpha+1}; && (8)\\ -p_{N-1}(x,y) &= d_{N,j}(x,y)(x - x_N)^{\alpha+1}. && (9) \end{aligned}$$

Thus, we obtain the conformality condition along the horizontal T-segment by the sum of the above equations

$$\sum_{i=1}^{N} d_{i,j}(x,y)(x - x_i)^{\alpha+1} \equiv 0. \tag{10}$$

Similarly, the conformality condition along the vertical T-segment is

$$\sum_{j=1}^{N} d_{i,j}(x,y)(y - y_j)^{\beta+1} \equiv 0, \tag{11}$$

where N denotes the number of the interior vertices on the vertical T-segment.

We regard the two equations as the first and the second class conformality conditions. Therefore, the global conformality conditions of a T-connected component are composed of the two classes of conformality equations along its all T-segments, and the global conformality conditions of different T-connected components are independent of each other. Similar to the spline spaces on the cross-cut partitions ([6]), the source cell has $(m+1)(n+1)$ degrees of freedom, and each cross-cut has a free smoothing cofactor. Thus, the dimensions of the spline spaces on the whole T-mesh are given by the following lemma.

Lemma 1 *Given a T-mesh $\mathcal{T}$ including C_h horizontal cross-cuts, C_v vertical cross-cuts and T different T-connected components $\mathcal{T}_i$, $i = 1, \cdots T$. Then the dimension of the spline space defined on $\mathcal{T}$ is*

$$\begin{aligned} & \dim \mathcal{S}(m, n, \alpha, \beta, \mathcal{T}) \\ = \ & (m+1)(n+1) + C_h(m+1)(n-\beta) + C_v(m-\alpha)(n+1) + \sum_{i=1}^{T} \dim \mathcal{T}_i, \end{aligned} \tag{12}$$

where $\dim \mathcal{T}_i$ denotes the dimension of the global conformality conditions corresponding to the i-th T-connected component.

If a T-connected component includes only one T-segment, the global conformality condition of the component is just one conformality condition of the first class or the second class. However, it is too difficult to discuss the dimension of the global conformality conditions of any T-connected component directly.

3.2 The dimensions of the two classes of conformality conditions

By Lemma 1, if we obtain the dimension of each T-connected component, then we obtain the dimension of whole spline space. Now we consider the two classes of conformality conditions including N vertices, and denote their dimensions by $\dim_1(N)$ and $\dim_2(N)$, respectively. We have the following results.

Lemma 2 *Let $d_i(x, y) \in \mathbb{P}_{m-\alpha-1,n-\beta-1}$ and $u_+ = \max(0, u)$. The dimension of the solution space of the system of equations*

$$\sum_{i=1}^{N} d_i(x, y)(x - x_i)^{\alpha+1} \equiv 0 \tag{13}$$

is $\dim_1(N) = (n-\beta)(N(m-\alpha) - (m+1))_+$; the dimension of the solution space of the system of equations

$$\sum_{j=1}^{N} d_j(x, y)(y - y_j)^{\beta+1} \equiv 0 \tag{14}$$

is $\dim_2(N) = (m-\alpha)(N(n-\beta) - (n+1))_+$.

Proof Since $d_i(x, y) \in \mathbb{P}_{m-\alpha-1,n-\beta-1}$, there exist $a^i_j(x) \in \mathbb{P}_{m-\alpha-1}$, $j = 0, \cdots, n-\beta-1$, such that

$$d_i(x, y) = a^i_0(x) + a^i_1(x)y + a^i_2(x)y^2 + \cdots + a^i_{n-\beta-1}(x)y^{n-\beta-1} = \sum_{j=0}^{n-\beta-1} a^i_j(x)y^j.$$

Then

$$\sum_{i=1}^{N} d_i(x,y)(x-x_i)^{\alpha+1} = \sum_{i=1}^{N} \sum_{j=0}^{n-\beta-1} a_j^i(x)(x-x_i)^{\alpha+1} y^j \equiv 0.$$

Since all coefficients of y^j equal to 0, we have $(n-\beta)$ independent systems of equations

$$\sum_{i=1}^{N} a_j^i(x)(x-x_i)^{\alpha+1} \equiv 0, \ j=0,\cdots,n-\beta-1.$$

Now we focus on the system of equations

$$\sum_{i=1}^{N} a^i(x)(x+x_i)^{\alpha+1} \equiv 0. \tag{15}$$

We can expand each $a^i(x)$ as

$$a^i(x) = c_0^i + c_1^i(x+x_i) + c_2^i(x+x_i)^2 + \cdots + c_{m-\alpha-1}^i(x+x_i)^{m-\alpha-1},$$

where $c_k^i \in \mathbb{R}, k=0,\cdots,m-\alpha-1$. Then

$$a^i(x)(x+x_i)^{\alpha+1} = c_0^i(x+x_i)^{\alpha+1} + c_1^i(x+x_i)^{\alpha+2} + \cdots + c_{m-\alpha-1}^i(x+x_i)^m.$$

Denote

$$c = (c_1,\cdots,c_N)^T,$$
$$c_i = (c_{m-\alpha-1}^i,\cdots,c_1^i,c_0^i)^T,$$
$$A = (A_1,\cdots,A_N),$$
$$A_i = \begin{pmatrix} 1 & & & \\ C_m^1 x_i & 1 & & \\ C_m^2 x_i^2 & C_{m-1}^1 x_i & \ddots & 1 \\ \vdots & \vdots & \cdots & \vdots \\ C_m^m x_i^m & C_{m-1}^{m-1} x_i^{m-1} & \cdots & C_{\alpha+1}^{\alpha+1} x_i^{\alpha+1} \end{pmatrix}_{(m+1)\times(m-\alpha)},$$

where $C_m^\alpha = \dfrac{m!}{\alpha!(m-\alpha)!}$. Then the systems of equations (15) is equivalent to the linear system

$$Ac = 0.$$

Similar to [2], A is in fact the matrix corresponding to Hermite interpolation at the points $x_1,\cdots,x_N$ with respect to the set of functions

$$1, C_m^1 x, C_m^2 x^2, \cdots, C_m^m x^m.$$

Since these functions are just constant multiples of the usual power functions, we conclude that

$$\text{rank}(A) = \min(m+1, N(m-\alpha)),$$

then the dimension of the solution space of Eq. (15) is $(N(m-\alpha)-(m+1))_+$.

Thus, the dimension of the solution space of Eq. (13) is

$$\dim_1(N) = (n-\beta)(N(m-\alpha)-(m+1))_+.$$

Similarly, the dimension of the solution space of Eq. (14) is

$$\dim_2(N) = (m-\alpha)(N(n-\beta)-(n+1))_+.$$

The proof is completed.

Denote

$$M_0 = \frac{m+1}{m-\alpha}, \quad N_0 = \frac{n+1}{n-\beta}.$$

From the proof, it can be found that M_0 and N_0 are two thresholds. This is an essential relation between the degree, the smoothness and the structure of the T-mesh, which guarantees the dimension formula. If the number of the interior vertices $N \leq M_0$, then the smoothing cofactors on the horizontal T-segment vanish. Similarly, $N \leq N_0$ means the smoothing cofactors on the vertical T-segment vanish. Consequently, the corresponding T-segment vanishes from the original T-mesh.

The corresponding matrix A shows that there are at least $(N - M_0)_+$ conformality cofactors $d_i(x, y)$ belonging to the basis of the solution space of Eq. (13), and $(N - N_0)_+$ conformality cofactors $d_j(x, y)$ belonging to the basis of the solution space of Eq. (14). Especially, when $M_0 = N_0 \leq 2$ (i.e., $m \geq 2\alpha + 1$ and $n \geq 2\beta + 1$), except any two conformality cofactors, the rest $N - 2$ conformality cofactors along the T-segment are independent of each other.

3.3 The dimensions of the spline spaces on T-mesh

By using the regular order of T-segments mentioned in Section 2, we can obtain the dimension of a T-connected component with some constraints as in the following lemma.

Lemma 3 *Let $\mathcal{T}_i$ be the i-th T-connected component. There are T_h^i horizontal T-segments, T_v^i vertical T-segments and V_i interior vertices in $\mathcal{T}_i$, and each T-segment L_j contains h_j vertices including $h_j^{(1)}$ mono-vertices and $h_j^{(2)}$ multi-vertices except two end vertices ($h_j = h_j^{(1)} + h_j^{(2)} + 2$). Suppose $h_j^{(1)} + 2 \geq M_0$ if L_j is a horizontal T-segment, and $h_j^{(1)} + 2 \geq N_0$ if L_j is a vertical T-segment. Then the dimension of $\mathcal{T}_i$ is*

$$\dim \mathcal{T}_i = V_i(m-\alpha)(n-\beta) - T_h^i(m+1)(n-\beta) - T_v^i(m-\alpha)(n+1). \tag{16}$$

Especially, the dimension of a T-connected component including only one single vertex is $(m-\alpha)(n-\beta)$.

Proof Let $N_i = T_h^i + T_v^i$.

Case 1 When $\mathcal{T}_i$ has no T-cycle, the N_i T-segments can be arranged in a regular order as

$$L_{N_i} \to L_{N_i-1} \to \cdots \to L_2 \to L_1.$$

Denote by t_j the number of interior multi-vertices on L_j, which are common vertices with the latter $j-1$ T-segments $L_{j-1}, \cdots, L_1$. It is clear that $t_1 = 0, t_j \leq h_j^{(2)}, j = 2, \cdots, N_i$.

Denote by $\dim L_j = \dim_1(h_j)$ the dimension corresponding to the conformality equations along L_j if L_j is a horizontal T-segment, and $\dim L_j = \dim_2(h_j)$ the dimension corresponding to the conformality equations along L_j if L_j is a vertical T-segment, $j = 1, \cdots, N_i$. The global conformality conditions are composed of N_i conformality equations along the N_i T-segments. Then the dimension of the global conformality conditions can be determined by the inverse of the regular order.

1) Let L_1 have its all degree of freedom $\dim L_1$.

2) For L_j, $j = 2, \cdots, N_i$, subtracted by the degree of freedom of the conformality cofactors at the t_j common vertices with $L_1, \cdots, L_{j-1}$, which are determined by the latter already, then L_j has its real degree of freedom $\dim L_j - t_j(m-\alpha)(n-\beta)$. Hence we have

$$\dim \mathcal{T}_i = \sum_{j=1}^{N_i} (\dim L_j - t_j(m-\alpha)(n-\beta)).$$

In fact, the correctness of the above procedure can be guaranteed by hypothesis $h_j^{(1)} + 2 \geq M_0$ and $h_j^{(1)} + 2 \geq N_0$. When L_j is a horizontal T-segment, by Lemma 2, even if the conformality cofactors at $h_j^{(2)}$ interior multi-vertices are determined by the latter conformality equations along $L_1, \cdots, L_{j-1}$, the rest $h_j^{(1)} + 2$ conformality cofactors have $(n-\beta)((h_j - h_j^{(2)})(m-\alpha) - (m+1))$ degrees of freedom, since $h_j^{(1)} + 2 \geq M_0$. Similarly, the j-th vertical T-segment has $(m-\alpha)((h_j - h_j^{(2)})(n-\beta) - (n+1))$ degrees of freedom by $h_j^{(1)} + 2 \geq N_0$.

It is easy to know

$$\sum_{j=1}^{N_i} (h_j - t_j) = V_i.$$

By Lemma 2, the dimension of the T-connected component can be calculated by

$$\begin{aligned} \dim \mathcal{T}_i &= \sum_{j=1}^{N_i} (h_j - t_j)(m-\alpha)(n-\beta) - T_h^i(m+1)(n-\beta) - T_v^i(m-\alpha)(n+1) \\ &= V_i(m-\alpha)(n-\beta) - T_h^i(m+1)(n-\beta) - T_v^i(m-\alpha)(n+1). \end{aligned} \quad (17)$$

Case 2 When $\mathcal{T}_i$ has T-cycles, each T-cycle can be dismissed by adding a virtual mono-vertex as shown in Fig. 2(c). Then the dimension can be calculated as case 1 without T-cycle. Note that each T-segment in a T-cycle has at least one multi-vertex except two end vertices (otherwise, it can be excluded from the T-cycle). We can choose one of them as L_1 in the regular order which has all its degree of freedom. Then the multi-vertices on L_1 have their degree of freedom since $h_j^{(1)} + 2 \geq M_0$ and $h_j^{(1)} + 2 \geq N_0$. Hence, it is enough to compensate the degree of freedom of the virtual mono-vertex, i.e., the virtual mono-vertex has its real degree of freedom. As a result, the dimension of $\mathcal{T}_i$ with T-cycles can also be calculated correctly by Eq. (17), when the virtual mono-vertices vanish again. The proof is completed.

Consequently, by Lemma 1 and Lemma 3, we obtain the dimension of the whole spline space $\mathcal{S}(m, n, \alpha, \beta, \mathcal{T})$ with constraints $h_j^{(1)} + 2 \geq M_0$ and $h_j^{(1)} + 2 \geq N_0$.

Theorem 1 *Given a T-mesh $\mathcal{T}$, which includes C_h horizontal cross-cuts, C_v vertical cross-cuts, T_h horizontal T-segments, T_v vertical T-segments and V interior vertices, the j-th T-segment L_j contains $h_j^{(1)}$ mono-vertices except two end vertices. Suppose $h_j^{(1)} + 2 \geq M_0$ if L_j is a horizontal T-segment, and $h_j^{(1)} + 2 \geq N_0$ if L_j is a vertical T-segment. Then the dimension of the spline space defined on $\mathcal{T}$ is*

$$\begin{aligned}\dim \mathcal{S}(m,n,\alpha,\beta,\mathcal{T}) &= (m+1)(n+1) + V(m-\alpha)(n-\beta) \\ &+(C_h - T_h)(m+1)(n-\beta) + (C_v - T_v)(n+1)(m-\alpha).\end{aligned} \tag{18}$$

At last, let F denote the number of cells and E the number of interior grid edges in $\mathcal{T}$. Then

$$E = E_h + E_v, \quad E_h = V + C_h - T_h, \quad E_v = V + C_v - T_v.$$

By Euler's formula,

$$F - E + V - 1 = 0.$$

It is easy to prove that the dimension formula (18) is equivalent to the formula given by Deng et al. in [1] when $M_0 = N_0 < 2$, i.e., $m \geq 2\alpha + 1$ and $n \geq 2\beta + 1$, which is given by the following theorem.

Theorem 2 ([1]) *Given a regular T-mesh and a corresponding spline space $\mathcal{S}(m,n,\alpha,\beta,\mathcal{T})$, suppose $m \geq 2\alpha + 1$ and $n \geq 2\beta + 1$. Then*

$$\begin{aligned}&\dim \mathcal{S}(m,n,\alpha,\beta,\mathcal{T}) \\ =\ & F(m+1)(n+1) - E_h(m+1)(\beta+1) - E_v(\alpha+1)(n+1) + V(\alpha+1)(\beta+1),\end{aligned} \tag{19}$$

where F is the number of cells in $\mathcal{T}$, E_h and E_v the number of interior horizontal edges and the number of interior vertical edges respectively, and V the number of interior vertices.

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